Probability Distributions

1 Univariate discrete distributions

Distribution	Probability mass function	Mean, variances
Uniform $\mathcal{U}(N)$	$f(x\|N) = \frac{1}{N+1}$, $x = 0, 1, 2, \ldots, N$.	$E(X) = \frac{N}{2}$ $Var[X] = \frac{N(N+2)}{12}$
Poisson $\mathcal{POI}(\lambda)$	$f(x\|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, \ldots.\quad \lambda > 0$.	$E(X) = \lambda$ $Var(X) = \lambda$
Multinomial $\mathcal{MN}(n,$ $p_1, \ldots, p_k)$	$f(x\|n, p_1, \ldots, p_k)$ $= \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$, $x_i = 0, 1, \ldots, n$. $0 \leq p_i \leq 1.\ \sum_{i=1}^k p_i = 1$.	$E(X_i) = np_i$ $Var(X_i) = np_i q_i$ $Cov(X_i, X_j) = -np_i p_j$ $q_i = 1 - p_i$
Binomial $\mathcal{BN}(n, p)$ $(= \mathcal{MN}(n, p, q))$	$f(x\|n, p) = \binom{n}{p} p^x q^{n-x}$, $q = 1 - p,\ x = 0, 1, \ldots, n$. $0 \leq p \leq 1$.	$E(X) = np$ $Var(X) = npq$
Bernoulli $\mathcal{BR}(p)$ $(= \mathcal{BN}(1, p))$	$f(x\|p) = p^x q^{1-x}$, $q = 1 - p,\ x = 0, 1$. $0 \leq p \leq 1$.	$E(X) = p$ $Var(X) = pq$
Negative Binomial $\mathcal{NB}(r, p)$	$f(x\|r, p) = \binom{r+x-1}{x} p^r q^x$, $q = 1 - p,\ x = 0, 1, \ldots.$ $0 \leq p \leq 1$.	$E(X) = \frac{rq}{p}$ $Var(X) = \frac{rq}{p^2}$
Geometric $\mathcal{GEO}(p)$ $(= \mathcal{NB}(1, p))$	$f(x\|p) = pq^x$, $q = 1 - p,\ x = 0, 1, \ldots.$ $0 \leq p \leq 1$.	$E(X) = \frac{q}{p}$ $Var(X) = \frac{q}{p^2}$
Hypergeometric $\mathcal{HG}(n, m, k)$	$f(x\|n, m, k) = \frac{\binom{m}{x}\binom{n-m}{k-x}}{\binom{n}{k}}$, $\max(0, m - n + k) \leq x$ $\leq \min(m, k).\ n, m, k \geq 0$.	$E(X) = \frac{km}{n}$ $Var(X)$ $= \frac{km(n-m)(n-k)}{n^2(n-1)}$

2 Univariate continuous distributions

Distribution	Probability density function	Mean, variances
Uniform $\mathcal{U}(a,b)$	$f(x\|a,b) = \frac{1}{b-a}$, $-\infty < a < x < b < \infty$.	$E(X) = \frac{a+b}{2}$ $Var(X) = \frac{(b-a)^2}{12}$
Normal $\mathcal{N}(\mu,\sigma^2)$	$f(x\|\mu,\sigma^2)$ $= (2\pi\sigma^2)^{-\frac{1}{2}} \exp -\frac{(x-\mu)^2}{2\sigma^2}$, $-\infty < \mu,\ x < \infty.\ \ \sigma > 0$.	$E(X) = \mu$ $Var(X) = \sigma^2$
Gamma $\mathcal{G}(\alpha,\beta)$	$f(x\|\alpha,\beta)$ $= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$, $x,\alpha,\beta > 0$.	$E(X) = \frac{\alpha}{\beta}$ $Var(X) = \frac{\alpha}{\beta^2}$
Exponential $\mathcal{EX}(\beta)$ $(= \mathcal{G}(1,\beta))$	$f(x\|\beta) = \beta \exp(-\beta x)$, $0 \le x < \infty.\ \ \beta > 0$.	$E(X) = \frac{1}{\beta}$ $Var(X) = \frac{1}{\beta^2}$
Chisquare $\chi^2(\nu)$ $(= \mathcal{G}\left(\frac{\nu}{2},\frac{1}{2}\right))$	$f(x\|\nu) = 2^{-\nu/2}\Gamma(\frac{\nu}{2})^{-1}$ $\times x^{\nu/2-1} \exp(-\frac{x}{2})$, $x,\nu > 0$.	$E(X) = \nu$ $Var(X) = 2\nu$
Inverted Gamma $\mathcal{IG}(\alpha,\beta)$ $(= \mathcal{G}(\alpha,\beta)^{-1})$	$f(x\|\alpha,\beta)$ $= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} \exp\left(-\beta x^{-1}\right)$, $x,\alpha,\beta > 0$.	$E(X) =$ $\frac{\beta}{\alpha-1}, (\alpha > 1)$ $Var(X) =$ $\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}, (\alpha > 2)$
Inverted chisquare Inv-$\chi^2(\nu)$ $(= \chi^2(\nu)^{-1})$	$f(x\|\nu) = 2^{-\nu/2}\Gamma(\frac{\nu}{2})^{-1}$ $\times x^{-(\frac{\nu}{2}+1)} \exp\left(-\frac{1}{2x}\right)$, $x,\nu > 0$.	$E(X) =$ $\frac{1}{\nu-2}, (\nu > 2)$ $Var(X) =$ $\frac{2}{(\nu-2)^2(\nu-4)}, (\nu > 4)$
Scaled inverted chisquare Inv-$\chi^2(\nu,s^2)$ $(= \mathcal{IG}(\frac{\nu}{2},\frac{\nu s^2}{2}))$	$f(x\|\nu,s^2) = \left(\frac{\nu s^2}{2}\right)^{-\nu/2} \Gamma(\frac{\nu}{2})^{-1}$ $\times x^{-(\frac{\nu}{2}+1)} \exp\left(-\frac{\nu s^2}{2x}\right)$, $x,\nu > 0$.	$E(X) =$ $\frac{\nu s^2}{\nu-2}, (\nu > 2)$ $Var(X) =$ $\frac{2\nu^2 s^4}{(\nu-2)^2(\nu-4)}, (\nu > 4)$

Distribution	Probability density function	Mean, variances
t $\mathcal{T}_\nu(\mu,\sigma^2)$	$f(x\|\nu,\mu,\sigma^2)=\Gamma\left(\frac{\nu+1}{2}\right)$ $\times\Gamma\left(\frac{\nu}{2}\right)^{-1}\Gamma\left(\frac{1}{2}\right)^{-1}(\nu\sigma^2)^{-\frac{1}{2}}$ $\times\left\{1+\frac{1}{\nu}\frac{(x-\mu)^2}{\sigma^2}\right\}^{-\frac{\nu+1}{2}}$, $-\infty<\mu,\ x<\infty.\ \nu,\sigma^2>0.$	$E(X)=\mu$ $(\nu>1)$ $Var(X)=\frac{\nu}{\nu-2}\sigma^2$ $(\nu>2)$
Cauchy $\mathcal{C}(\mu,\sigma^2)$ $(=\mathcal{T}_1(\mu,\sigma^2))$	$f(x\|\theta,\sigma)$ $=(\pi\sigma)^{-1}\left\{1+\frac{(x-\mu)^2}{\sigma^2}\right\}^{-1}$, $-\infty<x,\ \mu<\infty.\ \ \sigma^2>0.$	$E(X)=$ does not exist $Var(X)=$ does not exist
Beta $\mathcal{BE}(\alpha,\beta)$	$f(x\|\alpha,\beta)$ $=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0<x<1.\ \ \alpha,\beta>0.$	$E(X)=\frac{\alpha}{\alpha+\beta}$ $Var(X)$ $=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
F 分布 $\mathcal{F}(\nu_1,\nu_2)$	$f(x\|\nu_1,\nu_2)$ $=\frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}$ $\times x^{\frac{\nu_1-2}{2}}\left(1+\frac{\nu_1}{\nu_2}x\right)^{-\frac{\nu_1+\nu_2}{2}}$, $x,\nu_1,\nu_2>0.$	$E(X)=\frac{\nu_2}{\nu_2-2}$ $(\nu_2>2)$ $Var(X)=\frac{2\nu_2^2}{(\nu_2-2)^2}$ $\times\frac{(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)},(\nu_2>4)$
Pareto $\mathcal{PA}(\alpha,\beta)$	$f(x\|\alpha,\beta)=\alpha\beta^\alpha x^{-(\alpha+1)}$, $x>\beta.\ \ \alpha,\beta>0.$	$E(X)=\frac{\alpha\beta}{\alpha-1},\alpha>1$ $Var(X)=$ $\frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)},\alpha>2$
Lognormal $\mathcal{LN}(\mu,\sigma^2)$ $(=\ln\mathcal{N}(\mu,\sigma^2))$	$f(x\|(\mu,\sigma)=(2\pi\sigma^2)^{-\frac{1}{2}}$ $\times x^{-1}\exp-\frac{(\log(x)-\mu)^2}{2\sigma^2}$, $-\infty<\mu<\infty.\ \ x,\sigma^2>0.$	$E(X)=\exp\left(\mu+\frac{\sigma^2}{2}\right)$ $Var(X)$ $=\exp(2\mu+\sigma^2)$ $\times\{\exp(\sigma^2)-1\}$

3 Multivariate continuous distributions

Distribution	Probability density function	Mean, variances
Multivariate normal $\mathcal{N}(\mu, \Sigma)$	$f(x\|\mu,\Sigma) = (2\pi)^{-k/2}\|\Sigma\|^{-1/2}$ $\times \exp -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)$, $x = (x_1, \ldots, x_k)'$, $-\infty < x_i, \mu_i < \infty$. $\Sigma = \{\sigma_{ij}\}$: positive definite	$E(X_i) = \mu_i$ $Var(X_i) = \sigma_{ii}$ $Cov(X_i, X_j) = \sigma_{ij}$
Multivariate t $\mathcal{T}_\nu(\mu, \Sigma)$	$f(x\|\mu,\Sigma,\nu) = \Gamma(\frac{\nu+k}{2})\Gamma(\frac{\nu}{2})^{-1}$ $\times(\pi\nu)^{-\frac{k}{2}}\|\Sigma\|^{-\frac{1}{2}}\{1+$ $\frac{1}{\nu}(x-\mu)'\Sigma^{-1}(x-\mu)\}^{-\frac{\nu+k}{2}}$, $x = (x_1, \ldots, x_k)'$, $-\infty < x_i, \mu_i < \infty$. $\nu > 0$. $\Sigma = \{\sigma_{ij}\}$:positive definite	$E(X_i) = \mu_i$ $(\nu > 1)$ $Var(X_i) = \frac{\nu}{\nu-2}\sigma_{ii}$ $Cov(X_i, X_j)$ $= \frac{\nu}{\nu-2}\sigma_{ij}$ $(\nu > 2)$
Dirichlet $\mathcal{D}(\alpha_1, \ldots, \alpha_k)$	$f(x\|\alpha_1, \ldots, \alpha_k) =$ $\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_k)} x_1^{\alpha_1-1}\cdots x_k^{\alpha_k-1}$, $0 \le x_i \le 1$. $\sum_{i=1}^k x_i = 1$. $\alpha_i > 0$. $\alpha_0 = \sum_{i=1}^k \alpha_j$.	$E(X_i) = \frac{\alpha_i}{\alpha_0}$ $Var(X_i)$ $= \frac{\alpha_i(\alpha_0-\alpha_i)}{\alpha_0^2(\alpha_0+1)}$ $Cov(X_i, X_j)$ $= -\frac{\alpha_i\alpha_j}{\alpha_0^2(\alpha_0+1)}$
Wishart $\mathcal{W}(\alpha, \Sigma)$	$f(X\|\alpha,\Sigma) = \Gamma_k(\alpha)^{-1}\|\Sigma\|^{-\frac{\alpha}{2}}$ $\times\|X\|^{\frac{\alpha-(k+1)}{2}} \exp -\frac{\mathrm{tr}\,(\Sigma^{-1}X)}{2}$, X, Σ: $k \times k$ positive definite $\Gamma_k(\alpha) = 2^{\frac{\alpha k}{2}}\pi^{\frac{k(k-1)}{4}}$ $\times \prod_{i=1}^k \Gamma(\frac{\alpha+1-i}{2}), \alpha > k-1$.	$E(X_{ij}) = \alpha\sigma_{ij}$ $Var(X_{ij})$ $= \alpha(\sigma_{ii}\sigma_{jj} + \sigma_{ij}^2)$ $Cov(X_{ij}, X_{kl})$ $= \alpha(\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})$
Inverse Wishart $\mathcal{IW}(\alpha, \Sigma)$ $(= \mathcal{W}(\alpha, \Sigma)^{-1})$	$f(X\|\alpha,\Sigma) = \Gamma_k(\alpha)^{-1}\|\Sigma\|^{-\frac{\alpha}{2}}$ $\times\|X\|^{-\frac{\alpha+k+1}{2}} \exp -\frac{\mathrm{tr}\,(\Sigma^{-1}X^{-1})}{2}$, X, Σ: $k \times k$ positive definite $\Gamma_k(\alpha)$:same as above $\Sigma^{-1} = \{\sigma^{ij}\}$.	$E(X_{ij}) = \frac{\sigma^{ij}}{\alpha-k-1}$ $(\alpha > k+1)$ $Var(X_{ij})$: omitted $Cov(X_{ij}, X_{kl})$: omitted